$$V_{\beta_j}=\int_0^\infty [(W-2i\beta_j R^{-1})(u_j q_{2_j}+v_j q_{3_j}+w_j q_{4_j})$$

$$+q_{4_j}p_j+q_{1_j}w_j]\mathrm{d}y \Big/ \int_0^\infty (u_j q_{2_j}+v_j q_{3_j}+w_j q_{4_j})\,\mathrm{d}y \qquad \text{(A6)}$$

If we define

$$M_0=-v_1 Du_2-v_2 Du_1-i(\alpha_1+\alpha_2)u_1 u_2$$

$$-i(\beta_1 w_2 u_1+\beta_2 w_1 u_2) \qquad \text{(A7)}$$

$$M_{1,2}=-v_0 D\bar{u}_{2,1}-\bar{v}_{2,1}Du_0-i(\alpha_0-\alpha_2)u_0\bar{u}_{2,1}$$

$$+i(\beta_2 w_0\bar{u}_2-\beta_0\bar{w}_{2,1}u_0) \qquad \text{(A8)}$$

$$N_0=-v_1 Dv_2-v_2 Dv_1-i(\alpha_1 u_2+\beta_1 w_2)v_1-i(\alpha_2 u_1+\beta_2 w_1)v_2 \qquad \text{(A9)}$$

$$N_{1,2}=-v_0 D\bar{v}_{2,1}-\bar{v}_{2,1}Dv_0-i(\alpha_0 u_{2,1}v_0-\alpha_2 u_0\bar{v}_{2,1})$$

$$-i(\beta_0\bar{w}_{2,1}v_0-\beta_2 w_0\bar{v}_{2,1}) \qquad \text{(A10)}$$

$$P_0=-v_0 Dw_2-v_2 Dw_0-i(\beta_1+\beta_2)w_1 w_2$$

$$-i(\alpha_1 u_2 w_1+\alpha_2 u_1 w_2) \qquad \text{(A11)}$$

$$P_{1,2}=-v_0 D\bar{w}_{2,1}-\bar{v}_{2,1}Dw_0-i(\beta_0-\beta_2)w_0\bar{w}_{2,1}$$

$$+(\alpha_{2,1}u_0\bar{w}_{2,1}-\alpha_0\bar{u}_{2,1}w_0) \qquad \text{(A12)}$$

Then

$$F_j=\frac{M_j q_{2_j}+N_j q_{3_j}+P_j q_{4_j}}{\int_0^\infty (u_j q_{2_j}+v_j q_{3_j}+w_j q_{4_j})\,\mathrm{d}y} \quad j=0,1,2 \qquad \text{(A13)}$$

In the preceding quantities, bars denote complex conjugates.

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J80-025

NA

Effect of Shear Deformation and Rotatory Inertia on the Stability of Beck's and Leipholz's Columns

0330 8060 8070

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Introduction

STABILITY of nonconservative systems is discussed in the works of Bolotin[1] and Leipholz.[2] Stability of slender cantilever columns subjected to follower forces using the finite-element method has been reported in Refs. 3-5. In the present Note the effects of shear deformation and rotatory inertia on the stability of cantilever columns subjected to 1) a concentrated follower force, Beck's column, and 2) a uniformly distributed follower force, Leipholz's column, are studied with use of the finite-element method. The finite-element formulation including shear deformation and rotatory inertia is the same as that given in Ref. 6. As special cases, critical loads for slender columns are obtained from the present solution which agree very well with the published work.[5]

Finite-Element Formulation

The matrix equation governing the stability problem is obtained as[3]

$$\lambda^2[M]\{q\}-[K]\{q\}+Q[G^c]\{q\}+Q[G^{Nc}]\{q\}=0 \qquad (1)$$

where $[K]$, $[M]$, $[G^c]$ and $[G^{Nc}]$ are the global elastic stiffness, mass and geometric stiffness for the conservative part of the load, and the geometric stiffness matrix for the nonconservative part of the load, respectively. In Eq. (1), $\lambda^2=m\omega^2L^4/EI$, where m is the mass per unit length, ω is the circular frequency, L is the length of the column, E is the Young's modulus, and I is the moment of inertia. For Beck's column, $Q=PL^2/\pi^2EI$ and for Leipholz's column $Q=pL^3/\pi^2EI$, where P is the tip load and p is the distributed load per unit length.

The element stiffness matrix $[k]$, the mass matrix $[m]$, geometric stiffness matrices $[g^c]$ and $[g^{Nc}]$ are obtained using the usual procedure[7] from the expressions:

$$U=\frac{1}{2}\int_0^l \left[EI\psi_x^2+\frac{5}{6}\left(\frac{EA}{2(1+\nu)}\epsilon_{xz}^2\right)\right]\mathrm{d}x \qquad (2)$$

$$T=\frac{1}{2}\omega^2\int_0^l m[w^2+r^2(w_x+\gamma^2)^2]\,\mathrm{d}x \qquad (3)$$

$$W^c=\frac{P}{2}\int_0^l w_x^2\mathrm{d}x \quad \text{for Beck's column} \qquad (4)$$

$$=\frac{p}{2}\int_0^l (l-x)\,w_x^2\mathrm{d}x \quad \text{for Leipholz's column} \qquad (5)$$

Received June 12, 1979; revision reviewed Aug. 21, 1979. Copyright © American Institute of Aeronautics and Astronautics, Inc., 1979. All rights reserved.

Index categories: Structural Stability; Structural Dynamics; Vibration.

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Table 1 Convergence study for Beck's column, $L/r = 10$

Number of elements	Q_{cr} Linear γ	Q_{cr} Cubic γ
1	5.954	3.876
2	1.301	1.047
4	1.063	1.032
8	1.035	1.023
16	1.024	...

Table 2 Critical loads for Beck's column, 8-element solution (cubic γ)

L/r	Q_{cr}	λ^2_{cr}
10	1.023	46.1
15	1.400	71.7
25	1.742	97.6
50	1.949	114
100	2.010	119
1000	2.032	121

Table 3 Convergence study for Leipholz' problem: cubic γ; $L/r = 10$

Number of elements	Q_{cr}
1	8.167
2	2.140
4	1.839
8	1.813

Table 4 Critical loads for Leipholz's problem: 8-element solution (cubic γ)

L/r	Q_{cr}	λ^2_{cr}
10	1.813	44.5
15	2.636	70.9
25	3.406	97.6
50	3.871	114
100	4.012	119
1000	4.059	121

and

$$W^{Nc} = -P(w_x+\gamma)\,|_{x=l}\, w(l) \quad \text{for Beck's column} \tag{6}$$

$$= -p\int_0^l (w_x+\gamma)\, w\, dx \quad \text{for Leipholz's column} \tag{7}$$

In the above expression ψ_x is the curvature, A is the area of cross section, r is the radius of gyration, w is the lateral displacement, γ is the shear rotation, and ν is the Poisson's ratio. Suffix x denotes differentiation with respect to the axial coordinate x.

For the derivation of the element matrices, w is assumed as cubic in x with nodal parameters w and w_x at each note. For γ two distributions are used: 1) a linear distribution with γ as a nodal parameter, and 2) a cubic distribution with γ and γ_x as nodal parameters.

Critical loads Q_{cr} are obtained with use of the dynamic criterion where the first two frequencies coalesce.

Numerical Results

Critical loads for Beck's column and Leipholz's column are obtained for various L/r ratios.

Table 1 shows the convergence study for 1- to 16-element solutions with both linear and cubic γ distribution for $L/r = 10$. From this table it is evident that the cubic displacement distribution for γ gives very accurate results even with 4 elements. Table 2 gives the critical loads and frequencies of coalescence λ^2_{cr} corresponding to a model with 8 elements and cubic γ. The results for $L/r = 1000$ agrees with the value given in Ref. 5.

Table 3 gives the convergence study for the Leipholz's column with cubic γ. The 8-element model gives satisfactory results. Table 4 gives the critical loads and coalescence frequencies for Leipholz's column with 8 elements and cubic γ. Again, the results obtained in the present study for $L/r = 1000$ agree very well with those of Ref. 5.

Concluding Remarks

From the results obtained in the present study the following conclusions can be drawn: 1) a cubic γ distribution gives very accurate results compared to a linear γ distribution; 2) for small L/r the reduction in critical load is significant; and 3) except for very small L/r the coalescence frequencies for a given L/r are the same for both Beck's column and Leipholz's column.

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J80-026

Stress Analysis of a Plate Containing Two Circular Holes Having Tangential Stresses

80004

80008

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Introduction

In this Note an elastostatic problem is solved using the procedure of Kolosoff-Muskhelishvili complex stress functions[1] and the Schwarz alternating method[2] of successive relaxation for multiplying connected regions. Using this technique the problem of stresses in an infinite plate with two rigid circular inclusions are solved by the author.[3,4]

This method is used for finding the stress field induced in an isotropic homogeneous infinite plate with two circular

Received Feb. 8, 1978; revision received June 7, 1979.

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Index categories: Structural Statics; Structural Design.

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